Journal of Pure and Applied Algebra 32 (1984) 219-229
North-Holland

COHOMOLOGICAL BEHAVIOUR OF THE REDUCTION MODULO A PRIME OF $GL_3(Z)$

Christophe SOULÉ

Dépt. Mathématique et Informatique, Université Paris VII, 5è ét., Tour 45-55, 2 place Jussieu, 75251 Paris Cedex 05, France

Michishige TEZUKA

Department of Mathematics, Tokyo Institute of Technology, Ohokayama, Meguroku, Tokyo, Japan

Nobuaki YAGITA

Department of Mathematics, Musashi Institute of Technology, Tamazutumi, Setagayaku, Tokyo, Japan

Communicated by K.W. Gruenberg
Received 15 February 1983

Introduction

Let $GL_3(Z)$ be the group of 3 by 3 invertible matrices with integral coefficients, p a prime number, F_p the field with p elements, and

$$r_p : GL_3(Z) \to GL_3(F_p)$$

the reduction modulo p. The map r_p induces morphisms of cohomology groups (with integral coefficients)

$$r_p^* : H^*(GL_3(F_p)) \to H^*(GL_3(Z)).$$

The purpose of this paper is to describe completely r_p^*.

Actually a presentation of $H^*(GL_3(F_p))$ (resp. $H^*(GL_3(Z))$) is given in [1] and [5] (resp. [3]), and we give here an expression for the images of generators via r_p^*. In Section 0, we describe the cohomology of $GL_3(Z)$ and $GL_3(F_p)$. In Section 1, we prove that r_2^* is injective on 6-torsion. In Section 2, we study the reduction of r_p^* to the p-torsion of $H^*(GL_3(F_p))$. In Section 3, we study r_p^* on $H^*(GL_3(F_p), F_l)$, when $l \neq p$. We also compute $r_p^*(\tilde{c}_i)$, where $\tilde{c}_i \in H^{2i}(GL_3(F_p))$, $1 \leq i \leq 3$, are the Chern classes of the Brauer lifting of the standard representation of $GL_3(F_p)$.

0. Some known results

In this section, we sum up some of the results needed in the later sections. Let

$H^*(G)$ denote the cohomology ring of a discrete group G with coefficients Z. When $x \in H^*(G)$, we write $|x|$ the degree of x.

0.1. The cohomology of $\mathrm{SL}_3(Z)$ and $\mathrm{GL}_3(Z)$ can be computed completely by using the reduction theory of positive definite real quadratic forms.

Theorem 0.1 (cf. [3]). (i) *$H^*(\mathrm{GL}_3(Z))$ is killed by multiplication by* 12.

(ii) *Let G and G' be two cyclic group of order three in $\mathrm{GL}_3(Z)$ which are not conjugate to each other. Let ε (resp. ε') be a nontrivial element in $H^2(G)$ (resp. $H^2(G')$). The map*

$$H^*(\mathrm{GL}_3(Z))_{(3)} \to H^*(G)_{(3)} \oplus H^*(G')_{(3)}$$

is injective. Its image is generated by ε^2 and ε'^2.

(iii) *Let H and H' be two subgroups of $\mathrm{SL}_3(Z)$ isomorphic to the dihedral group $\mathscr{D}_4$ of eight elements and contained in $\Gamma_{M'}, \Gamma_O$ respectively* (*notations of* [3]). *Then the map*

$$H^*(\mathrm{SL}_3(Z)_{(2)} \to H^*(H)_{(2)} \oplus H^*(H')_{(2)}$$

is injective.

Furthermore $H^*(\mathrm{SL}_3(Z))_{(2)}$ is generated by elements $u_1, u_2, \ldots, u_7$ with $|u_1| = |u_2| = 3$, $|u_3| = |u_4| = 4$, $|u_5| = 5$, and $|u_6| = |u_7| = 7$.

0.2. Let U be the group of upper triangular matrices in $\mathrm{GL}_3(F_p)$. It is a p-Sylow subgroup of $\mathrm{GL}_3(F_p)$, so the map $H^*(\mathrm{GL}_3(F_p))_{(p)} \to H^*(U)_{(p)}$ is injective.

Theorem 0.2 [5]. (i) *For $p=2$ the ring $H^*(U)$ is generated by elements y_1, y_2, e, v with $|y_1| = |y_2| = 2$, $|e| = 3$, $|v| = 4$.*

The subring $H^*(\mathrm{GL}_3(F_2))_{(2)}$ is generated by $y_1 v$, $y_1^2 + y_2^2 + v$ and e.

(ii) *Modulo its nilpotent elements, the ring $H^*(\mathrm{GL}_3(F_3))_{(3)}$ is generated by elements $b_1, (y_1 v)^2, (y_2 v)^2, y_1 y_2 v$, and $y_1^6 + y_2^6 + v^2$ of respective degrees* 4, 16, 16, 10 *and* 12.

0.3. Quillen described $H^*(\mathrm{GL}_3(F_q), F_l)$ for any finite field F_q, where l is a prime not dividing q, and $n \geq 1$ an integer. In our case he gets

Theorem 0.3 [1]. (i) *There are ring isomorphisms*

$$H^*(\mathrm{GL}_3(F_p), F_3) = \begin{cases} F_3[\hat{c}_2] \otimes \Lambda(e_2) & \text{when } p \equiv 2 \pmod 3 \\ F_3[\hat{c}_1, \hat{c}_2, \hat{c}_3] \otimes \Lambda(e_1, e_2, e_3) & \text{when } p \equiv 1 \pmod 3 \end{cases}$$

with $|\hat{c}_i| = 2i$ and $|e_i| = 2i - 1$.

(ii) *The ring $H^*(\mathrm{GL}_3(F_p), F_2)$ is generated by elements $\hat{c}_1, \hat{c}_2, \hat{c}_3, e_1, e_2, e_3$ such that $|\hat{c}_i| = 2i$ and $|e_i| = 2i - 1$* (*for relations see* [1]).

1. The reduction modulo two

Theorem 1. *The homomorphism*

$$r_2^*: H^*(\mathrm{SL}_3(F_2))_{(l)} \to H^*(\mathrm{SL}_3(Z))$$

is injective when $l=2$ *or* 3, $*>0$.

Proof. For $l=2$ we look at the subgroup $H' \simeq \mathcal{D}_4$ of $\mathrm{SL}_3(Z)$ generated by

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to check that its image $\bar{H}'$ in $\mathrm{SL}_3(F_2)$ is still $\mathcal{D}_4$, so it is a 2-Sylow subgroup of $\mathrm{SL}_3(F_2)$.

Therefore the restriction map

$$H^*(\mathrm{SL}_3(F_2))_{(2)} \to H^*(\bar{H}')$$

is injective and the theorem comes from the commutative diagram

$$\begin{array}{ccc} H^*(\mathrm{SL}_3(F_2))_{(2)} & \longrightarrow & H^*(\mathrm{SL}_3(Z)) \\ \big\downarrow & & \big\downarrow \\ H^*(\bar{H}') & \xrightarrow{\ \sim\ } & H^*(H') \end{array}$$

For $l=3$, let $G \simeq Z/3Z$ be the subgroup of $\mathrm{SL}_3(Z)$ generated by

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that its image G in $\mathrm{SL}_3(F_2)$ is a 3-Sylow subgroup. The same argument as above shows that

$$r_2^*: H^*(\mathrm{SL}_3(F_2))_{(3)} \to H^*(\mathrm{SL}_3(Z))$$

is injective. □

2. The image of $H^*(\mathrm{GL}_3(F_p))_{(p)}$

We use the notation of Section 0.

Theorem 2. (i) *For* $p=3$ *we have*

$$r_3^*(y_1^6+y_2^6+v)=\varepsilon^6+\varepsilon'^6,$$

and the other generators of $H^*(\mathrm{GL}_3(F_3))_{(3)}$ *are mapped to zero by* r_3^*.

(ii) *For* $p=2$ *we have*

$$r_2^*(e)=u_2, \qquad r_2^*(y_1^2+y_2^2+v)=u_3, \quad \textit{and} \quad r_2^*(y_1 v)=u_7.$$

Proof. (i) Let G and G' be the cyclic subgroups of $\mathrm{SL}_3(Z)$ generated by

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

respectively. They are not conjugate in $\mathrm{SL}_3(Z)$, so the map

$$H^*(\mathrm{GL}_3(Z))_{(3)} \to H^*(G) \oplus H^*(G')$$

is injective (Theorem 0.1). The images $\bar{G}$ and $\bar{G}'$ in $\mathrm{SL}_3(F_3)$ are conjugate to the groups generated by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The commutative diagram

$$\begin{array}{ccc} H^*(\mathrm{GL}_3(F_3))_{(3)} & \longrightarrow & H^*(\mathrm{GL}_3(Z)) \\ \downarrow & & \downarrow \\ H^*(\bar{G}) \oplus H^*(\bar{G}') & \xrightarrow{\sim} & H^*(G) \oplus H^*(G') \end{array}$$

shows that it will be enough to study the restriction maps from $\mathrm{GL}_3(F_3)$ to $\bar{G}$ and $\bar{G}'$.

Let $H^*(G)=Z/3[\varepsilon]$ and $H^*(G')=Z/3[\varepsilon']$. Since U contains $\bar{G}$ and $\bar{G}'$, we can first study the map

$$H^*(U) \to H^*(\bar{G}) \oplus H^*(\bar{G}').$$

Using [5, (1.2) and (1.3)], we have $b^2 \mid \bar{G} = y_1^2 y_2^2 \mid \bar{G} = 0$, and we deduce that

$$y_1 \mid \bar{G} = \varepsilon, \qquad y_2 \mid \bar{G} = v \mid \bar{G} = b \mid \bar{G} = 0.$$

Similarly,

$$y_1 \mid \bar{G}' = \varepsilon', \qquad y_2 \mid \bar{G}' = v \mid \bar{G}' = b \mid \bar{G}' = 0.$$

We deduce from this that $r_3^*(y_1^6+y_2^6+v^2)=\varepsilon^6+\varepsilon'^6$ and that the other generators of $H^*(\mathrm{GL}_3(F_3))_{(3)}$ map to zero.

Notice that there are no nilpotents in $H^*(\mathrm{GL}_3(Z))_{(3)}$.

(ii) Let $H \subset \Gamma_O$ be the subgroup of $\mathrm{SL}_3(Z)$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and $H' \subset \Gamma_{M'}$ the group generated by

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using the fact that $H' \simeq \bar{H}' = U$, we get (cf. [5, Theorem 5.4])

$$y_1 v \,|\, H' = x_1 x_4, \qquad y_1^2 + y_2^2 + v \,|\, H' = x_2^2 + x_4 \quad \text{and} \quad e \,|\, H' = x_3,$$

where x_1, x_2, x_3, x_4 are the generators of $H^*(\mathscr{S}_4)$ given in [3].

From this it follows that

$$y_1 v \,|\, \Gamma_{M'} = z_3, \qquad y_1^2 + y_2^2 + v \,|\, \Gamma_{M'} = z_2 \quad \text{and} \quad e \,|\, \Gamma_{M'} = z_1$$

(notation of [3]).

To compute $H^*(\mathrm{SL}_3(F_2))_{(2)} \to H^*(\Gamma_O)_{(2)}$, denote by j_1 (resp. j_2) the inclusion of the group $Z/2Z$ generated by

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

into $\Gamma_{M'}$(resp. Γ_O), and by φ (resp. φ') the inclusion of $\Gamma_{M'}$(resp. Γ_O) into $\mathrm{SL}_3(Z)$. We have [3] $(\varphi' \cdot j_2)^* = (\varphi \cdot j_1)^*$. Composing this with r_2^*, we get

(a) $$y_1 v \,|\, Z/2Z = 0, \qquad y_1^2 + y_2^2 + v \,|\, Z/2Z = t^2 \quad \text{and} \quad e \,|\, Z/2Z = 0,$$

where $t \in H^2(Z/2Z)$ is the generator.

Let $\sigma : \mathscr{S}_4 \to Z/2Z$ be the signature morphism. We have $\sigma \cdot j_2 = \mathrm{id}$. Therefore, in $\mathrm{SL}_3(F_2)$, we get $\bar{\sigma} \cdot \bar{j}_2 = \mathrm{id}$. Furthermore $\bar{\Gamma}_O \simeq \mathscr{S}_3$, by [3, Lemma 0]. By [3, Lemma 8], the morphism

$$\bar{j}_2^{*-1} = \bar{\sigma}^* : H^*(Z/2Z) \to H^*(\mathscr{S}_3)_{(2)}$$

is an isomorphism. Moreover, if we call y_1' the generator of $H^2(\mathscr{S}_4)_{(2)} = Z/2Z$, we have that $\sigma^*(t) = y_1$ and $\sigma^* : H^*(Z/2Z) \to H^*(\Gamma_O)_{(2)}$ is injective. Therefore

$$r_2^* \simeq \sigma^* \bar{j}_2^* : H^*(\bar{\Gamma}_O)_{(2)} \to H^*(\Gamma_O)$$

is injective, and we have $\bar{\varphi}' \cdot \bar{j}_2 = r_2 \cdot \varphi' \cdot j_2$.

From the arguments above, we can evaluate

$$r_2^* \cdot \bar{\varphi}'^* = \sigma^* \cdot \bar{j}_2^* \cdot \bar{\varphi}'^* = \sigma^* \cdot j_2^* \cdot \varphi'^* \cdot r_2^* = \sigma^* \cdot j_1^* \cdot \varphi^* \cdot r_2^*.$$

From (a), we obtain

(b) $$y_1 v \,|\, \Gamma_O = 0, \qquad y_1^2 + y_2^2 + v \,|\, \Gamma_O = y_1^2 \quad \text{and} \quad e \,|\, \Gamma_O = 0.$$

Recall that the generators u_2, u_3, u_7 of $H^*(\mathrm{GL}_3(Z))_{(2)}$ are chosen such that $u_2 = z_1$, $u_3 = y_1^2 + z_2$ and $u_7 = z_3$ [3, Theorem 4(iv)].

The facts (a) and (b) imply

$$r_2^*(e) = u_2, \qquad r_2^*(y_1^2 + y_2^2 + v) = u_3 \quad \text{and} \quad r_2^*(y_1 v) = u_7. \qquad \square$$

3. The image of Chern classes

3.1. In this section, we fix a prime $l = 2, 3$ and a prime p different from l. We shall study the image via the reduction homomorphism $r_p^* : H^*(\mathrm{GL}_3(F_p)) \to H^*(\mathrm{GL}_3(Z))$ of some classes $\tilde{c}_i \in H^{2i}(\mathrm{GL}_3(F_p))_{(l)}$ defined as follows. Let $\bar{F}_p$ be an algebraic closure of F_p and $\varrho : \bar{F}_p^\times \to \mathbb{C}^\times$ a fixed embedding. When G is a finite group, we denote by $R_k(G)$ (resp. $R(G)$) the Grothendieck group of representations of G over a field k (resp. the complex field $\mathbb{C}$). To the embedding ϱ, we attach a Brauer lifting $\phi : R_k(G) \to R(G)$ for any finite extension k of F_p [2, 18.4]. By definition, $\tilde{c}_i \in H^{2i}(\mathrm{GL}_3(F_p))$ will be the Chern classes of the Brauer lifting of the natural representation of $\mathrm{GL}_3(F_p)$.

We also define $c_i \in H^{2i}(\mathrm{GL}_3(Z))_{(l)}$ to be the l-torsion part of the Chern classes of the embedding $\mathrm{GL}_3(Z) \to \mathrm{GL}_3(\mathbb{C})$.

Lemma 3.1. *We have* $r_p^*(\tilde{c}_i) = c_i$.

Proof. Let G be a subgroup of $\mathrm{GL}_3(Z)$ whose order is a power of l. We shall study the restriction of c_i and $r_p^*(\tilde{c}_i)$ to $H^{2i}(G)$. Since the cohomology of $\mathrm{GL}_3(Z)$ is detected by such groups (see Theorem 0.1), the lemma will follow from the equalities

$$c_i \mid G = r_p^*(\tilde{c}_i) \mid G.$$

Let K be a local field with characteristic 0 and residue field k, a finite extension of F_p such that the order of G divides the order of k. Let $\varrho : K \to \mathbb{C}$ be a fixed embedding of K into $\mathbb{C}$ and $\varrho : k^\times \to K^\times \to \mathbb{C}^\times$ the associate lifting of the units of k into $\mathbb{C}^\times$. (Remark that $\tilde{c}_i$ does not depend on the choice of this embedding.) Then the inclusion homomorphism $G \to \mathrm{GL}_3(Z) \to \mathrm{GL}_3(\mathbb{C})$ factors through $G \xrightarrow{j} \mathrm{GL}_3(K) \xrightarrow{\varrho} \mathrm{GL}_3(\mathbb{C})$. We have

$$c_i \mid G = j^* \varrho^*(c_i).$$

Let r_K be the decomposition homomorphism $R_K(G) \to R_k(G)$. Then we know by [2, 15.5] that r_k is an isomorphism. Denote by $\Phi : R_k(G) \to R_K(G)$ its inverse. We have $\phi = \varrho \cdot \Phi$, where ϱ is the embedding $R_K(G) \to R(G)$ defined by $\varrho([M]) = [M \otimes_K \mathbb{C}]$ (cf. [2, 18.4]). We denote by In the embedding $\mathrm{GL}_3(F_p) \to \mathrm{GL}_3(k)$. We have

$$r_p^*(\tilde{c}_i) \mid G = r_p^*(c_i(\varrho \cdot \Phi(\mathrm{In} \mid G)))$$

$$= c_i(\varrho \cdot \Phi(\mathrm{In} \cdot r_p | G)) = c_i(\varrho(\phi(r_K(j))))$$
$$= c_i(\varrho \cdot j) = c_i | G. \qquad \square$$

3.2. We shall find an expression of the Chern classes $c_i = r_p^*(\tilde{c}_i)$ in terms of the generators of $H^*(GL_3(Z))$. Let us fix some notations. The 3-torsion $H^*(GL_3(Z))_{(3)}$ of $H^*(GL_3(Z))$ is generated by classes ε^2 and ε'^2 defined in Theorem 0.1(ii). The 2-torsion $H^*(GL_3(Z))_{(2)}$ of $H^*(GL_3(Z))$ is generated by classes $u_1, u_2, \ldots, u_7$ in $H^*(SL_3(Z))_{(2)}$ (Theorem 0.1(iii)) and by the class $u_0 \in H^2(GL_3(Z))$ which is obtained from the determinant

$$\det^* : H^2(Z/2Z) \to H^2(GL_3(Z)).$$

Theorem 3.2. (i) *For $l = 3$, we have*

$$r_p^*(\tilde{c}_1) = r_p^*(\tilde{c}_3) = 0, \qquad r_p^*(\tilde{c}_2) = -\varepsilon^2 - \varepsilon'^2.$$

(ii) *For $l = 2$, we have*

$$r_p^*(\tilde{c}_1) = u_0, \qquad r_p^*(\tilde{c}_2) = -u_3 - u_4, \quad \textit{and} \quad r_p^*(\tilde{c}_3) = u_1^2 + u_2^2.$$

Proof. (i) We get $r_p^*(\tilde{c}_1) = r_p^*(\tilde{c}_3) = 0$ by noticing that $H^n(GL_3(Z))_{(3)} = 0$ when $n = 2$ and 6.

Let $\chi : G \to Z/3Z$ be an isomorphism and $\tilde{\chi} = \exp(2\pi i \chi/3) : G \to \mathbb{C}^\times$ the complex character attached to χ.

The generator $\varepsilon \in H^2(G)$ can be defined as the first Chern class of $\tilde{\chi}$. On the other hand a generator of G has eigenvalues 1, $\exp(2\pi i/3)$ and $\exp(4\pi i/3)$. So we get

$$c_2 | G = c_1(\tilde{\chi})c_1(\tilde{\chi}^{-1}) = -c_1(\chi)^2 = -\varepsilon^2.$$

The same argument gives $c_2 | G' = -\varepsilon'^2$.

(ii) Since the generator of $H^2(Z/2Z)$ is the first Chern class of the character $Z/2Z \to \mathbb{C}^\times$, we have

$$c_1 = c_1(\det) = u_0.$$

To evaluate c_2 and c_3, consider first the dihedral group of order eight $\mathcal{D}_4$. Its complex irreducible representations are the trivial representation, three nontrivial one-dimensional representations, and one irreducible representation φ of dimension two. Therefore any faithful representation $\psi : \mathcal{D}_4 \to SL_3(\mathbb{C})$ must be conjugate to $\varphi \oplus \det(\varphi)$. We get

$$c_2(\psi) = c_1(\varphi)c_1(\det \varphi) + c_2(\varphi) = c_1(\varphi)^2 + c_2(\varphi),$$
$$c_3(\varphi) = c_1(\varphi)c_2(\varphi).$$

Let a and b be generators of $\mathcal{D}_4$ submitted to the relations $a^4 = b^2 = (ab)^2 = 1$. We can realize φ by taking

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that $\det(a)=1$ and $\det(b)=-1$. If we use the notations of [3, Proposition 2(ii)], we have

$$c_1(\varphi)=c_1(\det\varphi)=x_2.$$

On the other hand, we have, from [3],

$$c_2(\varphi)=\lambda x_1^2+\mu x_2^2+\nu x_4$$

and we want to compute λ, μ and ν. Let χ be a character of order four of the group $Z/4Z=\langle a\rangle$ generated by a. We have $\varphi\,|\,\langle a\rangle=\chi\oplus\chi^{-1}$ and so $c_2(\varphi)\,|\,\langle a\rangle=-c_1(\chi)^2$.

Let $s\in H^2(\langle a\rangle)$ be a generator. Then we have $c_2(\varphi)\,|\,\langle a\rangle=-s^2$. As shown in [3],

$$x_1\,|\,\langle a\rangle=2s,\qquad x_2\,|\,\langle a\rangle=0\quad\text{and}\quad x_4\,|\,\langle a\rangle=s^2.$$

So we must have $\nu=-1$.

Consider the restriction of $c_2(\varphi)$ to $\langle a^2,b\rangle$. Let w_1' and $w_2':\langle a^2,b\rangle\to\mathbb{C}^\times$ be the characters such that $w_1'(a^2)=-1$, $w_1'(b)=1$, $w_2'(a^2)=1$, $w_2'(b)=-1$ and $w_1=c_1(w_1')$, $w_2=c_1(w_2')$. We know from [3] that $H^*(\langle a^2,b\rangle)$ is generated by the elements w_1, w_2 and a class $w_3\in H^3(\langle a^2,b\rangle)$ submitted to the relations

$$2w_1=2w_2=2w_3=w_3^2+w_1w_2(w_1+w_2)=0.$$

Since

$$a^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\quad\text{and}\quad b=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

we have

$$c_2(\varphi)\,|\,\langle a^2,b\rangle=c_1(w_1')c_1(w_1'w_2')=w_1^2+w_1w_2.$$

On the other hand, by [3], we know

$$x_1\,|\,\langle a^2,b\rangle=0,\qquad x_2\,|\,\langle a^2,b\rangle=w_2,\qquad x_4\,|\,\langle a^2,b\rangle=w_1^2+w_1w_2.$$

So we get $\mu=0$.

Finally we restrict $c_2(\varphi)$ to $\langle ab\rangle$. It is trivial that $c_2(\varphi)=0$. If $t\in H^2(\langle ab\rangle)$ is the generator, we have

$$x_1\,|\,\langle ab\rangle=x_2\,|\,\langle ab\rangle=t\quad\text{and}\quad x_4\,|\,\langle ab\rangle=0.$$

So we get $\lambda=\mu=0$.

In conclusion we have proved that $c_2(\varphi)=-x_4$ and for any faithfull representation $\psi:\mathscr{D}_4\to\mathrm{SL}_3(\mathbb{C})$ we have

$$c_2(\psi)=x_2^2-x_4,\qquad c_3(\psi)=x_2x_4.$$

We recalled in Theorem 0.1(iii) that $H^*(\mathrm{SL}_3(Z))_{(2)}$ is detected by a subgroup $H=\mathscr{D}_4$ contained in $\Gamma_O=\mathscr{S}_4$ and another group $H'=\mathscr{D}_4$ contained in $\Gamma_{M'}=\mathscr{S}_4$. We choose the inclusions $H\to\Gamma_O$ and $H'\to\Gamma_{M'}$ to be i_1 in the notations of [3, Proposition 3]. From [3, Theorem 4(iv)], we have

$$u_1\,|\,H=x_3,\qquad u_2\,|\,H=0,\qquad u_3\,|\,H=x_1^2,\qquad u_4\,|\,H=x_1^2+x_2^2+x_4,$$

$$u_5 \mid H = x_1x_3, \qquad u_6 \mid H = x_1x_4, \qquad u_7 \mid H = 0$$

and

$$u_1 \mid H' = u_4 \mid H' = u_5 \mid H' = u_6 \mid H' = 0,$$

$$u_2 \mid H' = x_3, \qquad u_3 \mid H' = x_4 + x_2^2, \qquad u_7 \mid H' = x_1x_4.$$

Put $c_2 = \lambda u_3 + \mu u_4$ and restrict to H and H'. We get

$$x_2^2 - x_4 = c_2(\psi) \mid H = \lambda x_1^2 + \mu(x_1^2 + x_2^2 + x_4),$$

$$x_2^2 - x_4 = c_2(\psi) \mid H' = \lambda(x_2^2 + x_4).$$

Therefore $c_2 = -u_3 - u_4$.

Put $c_3 = \lambda u_1^2 + \mu u_2^2 + \nu u_6 + \sigma u_7$ and restrict it to H. We get

$$x_4x_2 = c_3(\psi) \mid H = \lambda x_3^2 + \nu x_1x_4.$$

Since $x_3^2 = x_2x_4$, we get $\lambda = 1$ and $\nu = 0$.

Restrict to H'. We get

$$x_4x_2 = c_3(\psi) \mid H' = \mu x_3^2 + \sigma x_1x_4.$$

So we get $\mu = 1$ and $\sigma = 0$. Finally we have gotten $c_3 = u_1^2 + u_2^2$. □

3.3. The inclusion of groups $\mathrm{GL}_3(Z) \to \mathrm{GL}_3(\mathbb{C})$ induces a map between their classifying spaces

$$\varphi : \mathrm{BGL}_3(Z) \to \mathrm{BGL}_3(\mathbb{C})^{\mathrm{top}} = \mathrm{BU}_3.$$

Let $c_i \in H^{2i}(\mathrm{BU}_3)$, $1 \le i \le 3$, be the usual Chern classes. From Lemma 2.1 and Theorem 2.2 above we get

Corollary (see also [4]). *The kernel of*

$$\varphi^* : H^*(\mathrm{BU}_3) \to H^*(\mathrm{GL}_3(Z))$$

is generated by $2c_1$, $12c_2$ *and* $2c_3$.

3.4. Finally we shall describe the map

$$r_p^* : H^*(\mathrm{GL}_3(F_p), F_l) \to H^*(\mathrm{GL}_3(Z), F_l)$$

when $l \neq p$. When $x \in H^*(G)$, we denote by $\bar{x}$ its image in $H^*(G, F_l)$. We call $\beta_l : H^*(G, F_l) \to H^{*+1}(G)_{(l)}$ the Bockstein morphism attached to the exact sequence of coefficients

$$0 \to Z \xrightarrow{\times l} Z \to F_l \to 0.$$

By definition, [1], the classes $\hat{c}_i \in H^{2i}(\mathrm{GL}_3(F_p), F_l)$ satisfy $\hat{c}_i = \bar{\bar{c}}_i$. So $r_p^*(\hat{c}_i) = \overline{r_p^*(\bar{c}_i)} = \bar{c}_i$ is determined by Theorem 3.2 above.

To compute $r_p^*(e_i)$ we first remark that, by [1, Lemma 5], we have

$\beta_l(e_i) = ((p^i - 1)/l)c_i$. Therefore

$$\beta_l(r_p^*(e_i)) = \frac{p^i - 1}{l} c_i.$$

The map $\beta_3 : H^{2i-1}(\mathrm{GL}_3(Z), F_3) \to H^{2i}(\mathrm{GL}_3(Z))$ is injective, therefore the equality above is enough to compute $r_p^*(e_i)$. We get

Theorem 3.4. (i) *For* $l = 3$ *we have* $r_p^*(e_1) = r_p^*(e_3) = 0$ *and*

$$r_p^*(e_2) = \begin{cases} = 0 & \text{when } p \equiv 1 \text{ or } 8 \pmod 9, \\ \neq 0 & \text{when } p \equiv 2, 4, 5, 7 \pmod 9). \end{cases}$$

To compute $r_p^*(e_i)$ when $l = 2$ we use the same method as in Theorem 3.2. We just indicate the main steps. We have

$$H^*(\mathrm{GL}_3(Z), F_2) = H^*(Z/2Z, F_2) \otimes H^*(\mathrm{SL}_3(Z), F_2)$$

and the map

$$H^*(\mathrm{SL}_3(Z), F_2) \to H^*(H, F_2) \oplus H^*(H', F_2)$$

is injective. Recall that $H \simeq H' \simeq \mathscr{D}_4$.

Lemma 3.4. $H^*(\mathscr{D}_4, F_2) = F_2[s_1, s_2, w]/(s_1^2 + s_1 s_2)$, *with* $|s_1| = |s_2| = 1$, $|w| = 2$.

Assuming that $\mathscr{D}_4$ is generated by a and b, submitted to the relations $a^4 = b^2 = (ab)^2 = 1$, we take $s_1(a) = s_2(b) = 1$, $s_1(b) = s_2(a) = 0$. The element w is characterized by the equalities $\beta_2(w) = x_3$ and $\bar{x}_3 = ws_2$. We have $\bar{x}_1 = s_1^2$, $\bar{x}_2 = s_2^2$, $\bar{x}_3 = ws_2$, $\bar{x}_4 = w^2$.

Proposition 3.4. *The algebra* $H^*(\mathrm{GL}_3(Z), F_2)$ *is generated by* $u_0' \in H^1(Z/2Z, F_2)$ *and elements* $u_1', u_2', \ldots, u_6'$ *of respective degrees* 2, 2, 3, 3, 3, 3 *whose restrictions to* H *and* H' *are given by the following table*:

x	u_1'	u_2'	u_3'	u_4	u_5'	u_6'
$x\|H$	$w + s_2^2$	$s_1^2 + s_2^2$	ws_2	0	ws_1	0
$x\|H'$	$w + s_2^2$	$w + s_2^2$	0	ws_2	0	ws_1

To check this proposition we first prove that u_i' is in $H^*(\mathrm{GL}_3(Z), F_2)$, using [3, Theorem 4(ii)]. Then we show that when u_i is a generator of $H^*(\mathrm{GL}_3(Z))$, its reduction $\bar{u}_i$ is in the algebra generated by the elements u_i' (compute in H and H'). Finally we see that the elements $\beta_2(u_i)$ generate $\mathrm{Ker}(H^*(\mathrm{GL}_3(Z)) \xrightarrow{\times 2} H^*(\mathrm{GL}_3(Z)))$ as a module over $H^*(\mathrm{GL}_3(Z))$. □

Theorem 3.4. (ii) *For* $l = 2$ *and* $p \equiv 1 \pmod 4$ *we have*

$$r_p^*(e_i)=0, \quad 1\le i\le 3.$$

For $l=2$ *and* $p\equiv 3 \pmod 4$ *we have*

$$r_p^*(e_1)=u_0',$$

$$r_p^*(e_2)=u_3'+u_4',$$

$$r_p^*(e_3)=u_1'u_3'+u_1'u_4'.$$

Sketch of the proof. To get $r_p^*(e_1)$ we restrict this class to $Z/2Z$ and use [1, Proposition 3(ii)].

The representations $H\to GL_3(F_p)$ and $H'\to GL_3(F_p)$ are isomorphic to $\psi=\varphi\oplus\det\varphi$ as in Theorem 3.2. By [1, Proposition 3(ii)], we have

$$e_2(\psi)=e_2(\varphi) \quad\text{and}\quad e_3(\psi)=c_2(\varphi)e_1(\det\varphi)+e_2(\varphi)c_1(\det\varphi).$$

Using [1, Proposition 3(ii)], we get

$$e_1(\varphi)=e_1(\det\varphi)=\tfrac{1}{2}(p-1)s_2.$$

By restricting $e_2(\varphi)$ to the subgroups $\langle a\rangle,\langle a^2,b\rangle$ and $\langle ab\rangle$ of $\mathscr{Q}_4$ we get $e_2(\varphi)=\frac{1}{2}(p-1)ws_2$. We deduce

$$e_2(\psi)=\tfrac{1}{2}(p-1)ws_2 \quad\text{and}\quad e_3(\psi)=\tfrac{1}{2}(p-1)(w^2s_2+ws_2^3).$$

Since $e_i(\psi)=r_p^*(e_i)\,|\,H=r_p^*(e_i)\,|\,H'$, these relations determine $r_p^*(e_i)$. □

References

[1] D. Quillen, On the cohomology and K-theory of the general linear group over a finite field, Annals of Math 96 (1972) 552–586.

[2] J.-P. Serre, Représentations linéaires des groupes finis, 2nd ed. (Hermann, Paris, 1972).

[3] C. Soulé, The cohomology of $SL_3(Z)$, Topology 17 (1978) 1–22.

[4] C. Soulé, Classes de torsion dans la cohomologie des groupes arithmétiques, C.R. Acad. Sci. Paris (Série A) 284, 1010.

[5] M. Tezuka and N. Yagita, The mod p cohomology ring of $GL_3(F_p)$, J. of Algebra, 81 (1983) 295–303.